

A Note on Approximating 2-Transmitters

Saeed Mehrabi¹, Abbas Mehrabi²

¹ Cheriton School of Computer Science, University of Waterloo, Waterloo, Canada.
 smehrabi@uwaterloo.ca

² School of Information and Communications, Gwangju Institute of Science and
 Technology, Gwangju, South Korea. mehrabi@gist.ac.kr

Abstract. A k -transmitter in a simple orthogonal polygon P is a mobile guard that travels back and forth along an orthogonal line segment s inside P . The k -transmitter can see a point $p \in P$ if there exists a point $q \in s$ such that the line segment pq is normal to s and pq intersects the boundary of P in at most k points. In this paper, we give a 2-approximation algorithm for the problem of guarding a monotone orthogonal polygon with the minimum number of 2-transmitters.

1 Introduction

In the standard version of the art gallery problem, introduced by Klee in 1973 [10], we are given a simple polygon P in the plane and the goal is to guard P by a set of point guards. That is, we need to find a set of point guards such that every point in P is seen by at least one of the guards, where a guard g sees a point p if and only if the segment gp is contained in P . Chvátal [1] proved that $\lfloor n/3 \rfloor$ point guards are always sufficient and sometimes necessary to guard a simple polygon with n vertices. The art gallery problem is known to be NP-hard on arbitrary polygons [8], orthogonal polygons [11] and even monotone polygons [7]. Eidenbenz [4] proved that the art gallery problem is APX-hard on simple polygons and Ghosh [5] gave an $O(\log n)$ -approximation algorithm that runs in $O(n^4)$ time on simple polygons. Krohn and Nilsson [7] gave a constant-factor approximation algorithm on monotone polygons. They also gave a polynomial-time algorithm for the orthogonal art-gallery problem that computes a solution of size $O(OPT^2)$, where OPT is the cardinality of an optimal solution.

Many variants of the art gallery problem have been studied. Katz and Morgenstern [6] introduced a variant of this problem in which k -transmitters are used to guard orthogonal polygons. A k -transmitter T , where $k \geq 0$, is a point guard that travels back and forth along an orthogonal line segment inside an orthogonal polygon P . A point p in P is visible to T , if there is a point q on T such that the line segment pq is normal to T and it intersects the boundary of P in at most k points. In the *Minimum k -Transmitters (MkT)* problem, the objective is to guard P with the minimum number of k -transmitters. Katz and Morgenstern introduced the MkT problem for only $k = 0$ (we remark that 0-transmitters are called *sliding cameras* in [6]). They first considered a restricted version of

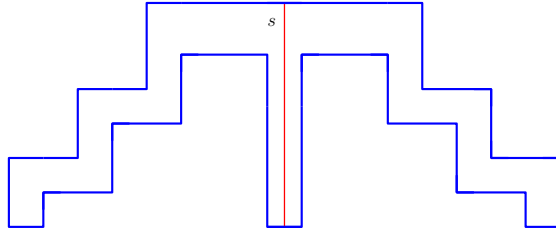


Fig. 1: A monotone orthogonal polygon P that can be guarded by a single 2-transmitter s while five 0-transmitters are required to guard P entirely. This example can be extended to show that the exact algorithm of de Berg et al. [2] for the M0T problem does not provide any constant-factor approximation to an exact solution for the M2T problem on P .

the problem, where only vertical 0-transmitters are allowed, and solved this restricted version in polynomial time for simple orthogonal polygons. When both vertical and horizontal 0-transmitters are allowed (i.e., the M0T problem), they gave a 2-approximation algorithm on monotone orthogonal polygons, which was later improved by the $O(n)$ -time exact algorithm of de Berg et al. [2]. Durocher and Mehrabi [3] showed that the M0T problem is NP-hard when P is allowed to have holes. Mahdavi et al. [9] proved that the problem of guarding an orthogonal polygon with k -transmitters so as to minimize the total length of line segments along which k -transmitters travel is NP-hard for any fixed $k \geq 2$, and gave a 2-approximation algorithm for this problem. To our knowledge, the complexity of the MkT problem is open on simple orthogonal polygons for any fixed $k \geq 0$.

We remark that the exact algorithm of de Berg et al. [2] for the M0T problem on monotone orthogonal polygons does not provide any constant-factor approximation algorithm for the M2T problem. Figure 1 shows a polygon P for which five 0-transmitters are required, but P can be guarded with only one 2-transmitter. Note that the example can be extended to show that an exact solution for the M0T problem does not provide any constant-factor approximation to that of the M2T problem.

Our Result. In this paper, we give a polynomial-time 2-approximation algorithm for the M2T problem on simple and monotone orthogonal polygons. Some preliminaries are given in Section 2. We then present our 2-approximation algorithm in Section 3 and conclude the paper in Section 4.

2 Preliminaries

Throughout this paper, let P be a simple and x -monotone orthogonal polygon with n vertices. A vertex u of P is called *convex* (resp., *reflex*), if the angle at u that is interior to P is 90° (resp., 270°). We denote the leftmost and rightmost vertical edges of P (that are unique) by $\text{leftEdge}(P)$ and $\text{rightEdge}(P)$, respectively. Let $V_P = \{e_1 = \text{leftEdge}(P), e_2, \dots, e_m = \text{rightEdge}(P)\}$, for some

$m > 0$, be the set of vertical edges of P ordered from left to right. Let P_i^+ (resp., P_i^-), for some $1 \leq i \leq m$, denote the subpolygon of P that lies to the right (resp., to the left) of the vertical line through e_i .

Let s be an orthogonal line segment in P . We denote the left endpoint and the right endpoint of s by $\text{left}(s)$ and $\text{right}(s)$, respectively. If s is vertical, we define its left and right endpoints to be its upper and lower endpoints, respectively. Moreover, we denote the k -transmitter that travels along s by $s^{(k)}$. For a k -transmitter t in P , we define the *visibility polygon* of t as the maximal subpolygon $\text{Vis}(t)$ of P such that every point in $\text{Vis}(t)$ is guarded by t .

For each reflex vertex v of P , extend the edges incident to v inward until they hit the boundary of P . Let $C(P)$ be the set of all maximal line segments in P that are obtained in this way. A *feasible solution* for the M2T problem is a set M of 2-transmitters that guards the entire polygon P . A feasible solution M is *optimal* (or, *exact*) if $|M| \leq |S'|$, for all feasible solutions S' . We say that a feasible solution M for the M2T problem is in *standard form* if and only if $M \subseteq C(P)$ and every vertical 2-transmitter in M is *vertically maximal*; that is, it extends as far upwards and downwards as possible.

Lemma 1. *There exists an optimal solution OPT^* for the M2T problem on P that is in standard form.*

Proof. Take any optimal solution OPT for the M2T problem on P . First, for each line segment $s \in OPT$ that is not aligned with an edge of P , move s vertically up or down, or horizontally to the left or right until it hits an edge of P . Next, for every line segment $s' \in OPT$ that is not maximal, replace s' with the maximal line segment in P that is aligned with s' . Set $OPT^* := OPT$. Clearly, OPT^* is a feasible solution for the M2T problem and every line segment in OPT^* is maximal and aligned with an edge of P . So, $OPT^* \subseteq C(P)$. Since $|OPT^*| \leq |OPT|$, we conclude that OPT^* is an optimal solution for the M2T problem that is in standard form. This completes the proof of the lemma. \square

For a horizontal line segment $t \in P$ and any $k > 0$, the visibility polygon of a 0-transmitter that travels along t is the same as that of a k -transmitter that travels along t . We state and prove this observation more formally.

Lemma 2. *Let t be a horizontal line segment in P . Then, $\text{Vis}(t^{(0)}) = \text{Vis}(t^{(k)})$ for any $k > 0$.*

Proof. It is clear that any point in P that is visible to $t^{(0)}$ is also visible to $t^{(k)}$ and so $\text{Vis}(t^{(0)}) \subseteq \text{Vis}(t^{(k)})$. Now, let p be a point in P that is visible to $t^{(k)}$. Since t is horizontal and P is an x -monotone orthogonal polygon, we conclude that the line segment pq does not intersect the boundary of P , where q is the projection of p onto t . This means that p is also visible to $t^{(0)}$ and therefore, $\text{Vis}(t^{(k)}) \subseteq \text{Vis}(t^{(0)})$. This completes the proof of the lemma. \square

3 A 2-Approximation Algorithm

In this section, we give our 2-approximation algorithm for the M2T problem on monotone orthogonal polygons. Recall that in the M2T problem, the objective is to guard the polygon P with minimum number of 2-transmitters, where a 2-transmitter can be either horizontal or vertical. For a point $p \in P$, let $L(p)$ denote the vertical line through p . We say that a horizontal 2-transmitter in P is *rightward maximal* if it extends as far to the right as possible.

The algorithm initially guards a leftmost portion of the polygon P by two 2-transmitters with different orientations, and then will guard the remaining part of P recursively. The order of the two initial 2-transmitters is determined by whether locating first a vertical 2-transmitter and then a horizontal one would guard a larger portion of P than locating first a horizontal 2-transmitter and then a vertical one. In the following, we describe the algorithm more formally.

Algorithm. Let s_v be the rightmost maximal vertical 2-transmitter in P such that every point of P that is to the left of s_v is seen by s_v ; let p be the leftmost point of P that is not seen by s_v . Moreover, let s_h be the rightward maximal horizontal 2-transmitter in P such that $\text{left}(s_h)$ lies on $L(p)$. Clearly, $\text{right}(s_h)$ lies on a vertical edge e_i of P . Observe that P_i^- is entirely guarded by s_v and s_h . Given P , we define $\text{vhFinder}(P)$ as a method that computes s_v and s_h as described above and returns the triple (s_v, s_h, e_i) . Note that $\text{vhFinder}(P)$ guards P_i^- by first locating a vertical 2-transmitter and then a horizontal one from left to right. We next consider the other case.

Let s'_h be the rightward maximal horizontal 2-transmitter in P such that every point of P that is to the left of $L(\text{right}(s'_h))$ is seen by s'_h . Suppose that $\text{right}(s'_h)$ lies on some vertical edge e_ℓ ($1 \leq \ell \leq m$) of P . Let s'_v be the rightmost maximal vertical 2-transmitter in P such that every point of P that lies between $L(\text{right}(s'_h))$ and s'_v is guarded by s'_v . Moreover, let p' be the leftmost point of P_ℓ^+ that is not seen by s'_v ; clearly, p' lies on a vertical edge e_j ($1 \leq j \leq m$) of P . Observe that s'_h and s'_v guard P_j^- entirely. We now define $\text{hvFinder}(P)$ as a method that computes s'_h and s'_v as described above and returns the triple (s'_h, s'_v, e_j) .

The algorithm is shown in Algorithm 1. In the first step of the algorithm, we remove from $C(P)$ those line segments whose visibility polygon is a subset of the union of the visibility polygons of all other line segments in $C(P)$. Then, in a while-loop, we iteratively (i) compute the pairs of 2-transmitters $\{s_v, s_h\}$ and $\{s'_h, s'_v\}$ using the methods $\text{vhFinder}(P)$ and $\text{hvFinder}(P)$, respectively, and then (ii) update P depending on whether $i > j$ (i.e., the 2-transmitters $\{s_v, s_h\}$ guard a larger portion of P than $\{s'_h, s'_v\}$) or $j \geq i$ (i.e., the 2-transmitters $\{s'_h, s'_v\}$ guard a larger portion of P than $\{s_v, s_h\}$). We remark here that by Lemma 1, we can assume that both methods $\text{vhFinder}(P)$ and $\text{hvFinder}(P)$ select the 2-transmitters from the set $C(P)$. When P is entirely guarded, we return the set S of 2-transmitters.

Algorithm 1 APPROXIMATE2TRANSMITTERS(P)

```

1: for each line segment  $s \in C(P)$  do
2:   if  $\text{Vis}(s) \subseteq \bigcup_{s' \in C(P) \setminus \{s\}} \text{Vis}(s')$  then
3:      $C(P) \leftarrow C(P) \setminus \{s\}$ ;
4:  $S \leftarrow \emptyset$ ;
5: while  $P \neq \emptyset$  do
6:    $(s_v, s_h, e_i) \leftarrow \text{vhFinder}(P)$ ;  $\triangleright \{s_v, s_h\} \subseteq C(P)$ 
7:    $(s'_h, s'_v, e_j) \leftarrow \text{hvFinder}(P)$ ;  $\triangleright \{s'_h, s'_v\} \subseteq C(P)$ 
8:   if  $i > j$  then
9:      $S \leftarrow S \cup \{s_v, s_h\}$ ;
10:     $P \leftarrow P_i^+$ ;
11:   else
12:      $S \leftarrow S \cup \{s'_h, s'_v\}$ ;
13:     $P \leftarrow P_j^+$ ;
14: return  $S$ ;

```

Analysis. We first note that by Lemma 1, we can assume that the four 2-transmitters computed by $\text{vhFinder}(P)$ and $\text{hvFinder}(P)$ are always in standard form. That is, we restrict our attention to the line segments in $C(P)$ when computing the set S . To see the approximation factor of the algorithm, let P_1, P_2, \dots, P_k be the partition of P into k subpolygons ordered from left to right such that the subpolygon P_i is guarded in the i th iteration of the while-loop. More precisely, P_i is the subpolygon of P that is cut out from P in the i th iteration of the while-loop of the algorithm. It is clear that Algorithm 1 locates at most $2k$ 2-transmitters to guard P entirely; that is, $|S| \leq 2k$. In the following, we show that $|OPT| \geq k$ for any optimal solution OPT for the M2T problem on P .

Lemma 3. *Let OPT be an optimal solution for the M2T problem on P . Then, $|OPT| \geq k$.*

Proof. By Lemma 1, we assume that OPT is in standard form; that is, $OPT \subseteq C(P)$ and every vertical 2-transmitter in OPT is vertically maximal. Consider the partition $T = \{P_1, P_2, \dots, P_k\}$ of P induced by the recursive steps of the algorithm, and let s be a horizontal line segment in P . We say that s *originates* from P_j , for some $1 \leq j \leq k$, if $\text{left}(s)$ lies inside P_j . Suppose for a contradiction that $|OPT| < k$. Then, there must be a subpolygon $P_i \in T$ such that neither a vertical 2-transmitter of OPT lies in P_i nor a horizontal 2-transmitter of OPT originates from P_i . We then must have one of the followings (w.l.o.g., we assume that Algorithm 1 located the pair $\{s_v, s_h\}$ in P_i):

- There exists at least one horizontal 2-transmitter in OPT that intersects $\text{leftEdge}(P_i)$ (and, therefore its left endpoint lies to the left of $\text{leftEdge}(P_i)$). Let s_h^* be the rightward maximal horizontal 2-transmitter among all such 2-transmitters. Clearly, s_h^* does not see P_i entirely because then $\text{hvFinder}(P)$

would have selected the portion of s_h^* that lies in P_i along with the vertical line segment s'_v and so P_i would have been extended further to the right. Now, let $P'_i := P_i \setminus \text{Vis}(s_h^*)$. Since s_h^* is rightward maximal and there is no horizontal 2-transmitter of OPT that is originated from P_i , we conclude that no horizontal 2-transmitter in P sees a point in P'_i . Therefore, there must a vertical 2-transmitter s_v^* that guards P'_i and that s_v^* lies to the left of $\text{leftEdge}(P_i)$ or to the right of $\text{rightEdge}(P_i)$ (recall that there is no vertical 2-transmitter of OPT inside P_i). (i) If s_v^* lies to the right of $\text{rightEdge}(P_i)$, then our algorithm would have added s_v^* and the portion of s_h^* that lies in P_i into S and so P_i would have been extended further to the right — a contradiction. (ii) If s_v^* lies to the left of $\text{leftEdge}(P_i)$, then we observe that s_v^* and s_h (i.e., the horizontal 2-transmitter located in P_i by our algorithm) would together guard P_i entirely. This means that $\text{Vis}(s_v) \subseteq (\text{Vis}(s_v^*) \cup \text{Vis}(s_h))$ and so s_v should have been removed from $C(P)$ in the first step of the algorithm — a contradiction.

- There is no horizontal 2-transmitter of OPT intersecting $\text{leftEdge}(P_i)$. This means that no point inside P_i is seen by a horizontal 2-transmitter in P_i . Moreover, since no vertical 2-transmitter of OPT lies in P_i , we conclude that P_i is guarded by a set $M \subseteq OPT$ of *only-vertical* 2-transmitters that lie to the left of $\text{leftEdge}(P_i)$ or to the right of $\text{rightEdge}(P_i)$. That is, $P_i \subseteq \bigcup_{s_j \in M} \text{Vis}(s_j)$. But, this means that $\text{Vis}(s_v) \subseteq \bigcup_{s_j \in M} \text{Vis}(s_j)$, which is a contradiction because then s_v should have been removed from $C(P)$ in the first step of the algorithm.

By the two cases above, we conclude that $|OPT| \geq k$. This completes the proof of the lemma. \square

Each call to methods $\text{vhFinder}(P)$ and $\text{hvFinder}(P)$ is completed in polynomial time. Moreover, the while-loop of Algorithm 1 terminates after at most m iterations (recall that m is the number of the vertical edges of P) because at least one new vertical edge of P is guarded at each iteration. Therefore, Algorithm 1 runs in polynomial time. Therefore, by Lemma 3 and the fact that $|S| \leq 2k$, we have the main result of this paper:

Theorem 1. *There exists a polynomial-time 2-approximation algorithm for the M2T problem on monotone orthogonal polygons.*

4 Conclusion

In this paper, we gave a polynomial-time 2-approximation algorithm for the M2T problem on monotone orthogonal polygons. The complexity of the problem remains open on simple orthogonal polygons. Similar to Katz and Morgenstern [6], it might be interesting to first consider the problem with only-vertical 2-transmitters.

References

1. Vasek Chvatal. A combinatorial theorem in plane geometry. *Journal of Combinatorial Theory, Series B*, 18:39–41, 1975.
2. Mark de Berg, Stephane Durocher, and Saeed Mehrabi. Guarding monotone art galleries with sliding cameras in linear time. In *proceedings of the 8th International Conference on Combinatorial Optimization and Applications (COCOA 2014)*, Wailea, Maui, HI, USA, December 19-21., pages 113–125, 2014.
3. Stephane Durocher and Saeed Mehrabi. Guarding orthogonal art galleries using sliding cameras: Algorithmic and hardness results. In *proceedings of the 38th International Symposium on Mathematical Foundations of Computer Science (MFCS 2013)*, Klosterneuburg, Austria, August 26-30., pages 314–324, 2013.
4. Stephan Eidenbenz. Inapproximability results for guarding polygons without holes. In *Proceedings of the 9th International Symposium Algorithms and Computation (ISAAC 1998)*, Taejon, Korea, December 14-16., pages 427–436, 1998.
5. Subir K. Ghosh. Approximation algorithms for art gallery problems in polygons. *Disc. App. Math.*, 158(6):718–722, 2010.
6. Matthew J. Katz and G. Morgenstern. Guarding orthogonal art galleries with sliding cameras. *Inter. J. Comp. Geom. & App.*, 21(2):241–250, 2011.
7. Erik Krohn and Bengt J. Nilsson. Approximate guarding of monotone and rectilinear polygons. *Algorithmica*, 66(3):564–594, 2013.
8. D. T. Lee and Arthur K. Lin. Computational complexity of art gallery problems. *IEEE Transactions on Information Theory*, 32(2):276–282, 1986.
9. Salma Sadat Mahdavi, Saeed Seddighin, and Mohammad Ghodsi. Covering orthogonal polygons with sliding k-transmitters. In *proceedings of the 26th Canadian Conference on Computational Geometry, CCCG 2014, Halifax, Nova Scotia, Canada.*, 2014.
10. Joseph O’Rourke. *Art Gallery Theorems and Algorithms*. The International Series of Monographs on Computer Science. Oxford University Press, New York, NY, 1987.
11. Dietmar Schuchardt and Hans-Dietrich Hecker. Two NP-hard art-gallery problems for ortho-polygons. *Mathematical Logic Quarterly*, 41(2):261–267, 1995.